

Obstructing Visibilities with One Obstacle^{*}

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Abstract. Obstacle representations of graphs have been investigated quite intensely over the last few years. We focus on graphs that can be represented by a single obstacle. Given a (topologically open) non-self-intersecting polygon C and a finite set P of points in general position in the complement of C , the *visibility graph* $G_C(P)$ has a vertex for each point in P and an edge pq for any two points p and q in P that can see each other, that is, $\overline{pq} \cap C = \emptyset$. We draw $G_C(P)$ straight-line and call this a *visibility drawing*. Given a graph G , we want to compute an obstacle representation of G , that is, an obstacle C and a set of points P such that $G = G_C(P)$. The complexity of this problem is open, even when the points are exactly the vertices of a simple polygon and the obstacle is the complement of the polygon—the *simple-polygon visibility graph problem*.

There are two types of obstacles; *outside* obstacles lie in the unbounded component of the visibility drawing, whereas *inside* obstacles lie in the complement of the unbounded component. We show that the class of graphs with an inside-obstacle representation is incomparable with the class of graphs that have an outside-obstacle representation. We further show that any graph with at most seven vertices has an outside-obstacle representation, which does not hold for a specific graph with eight vertices. Finally, we show NP-hardness of the *outside-obstacle graph sandwich problem*: given graphs G and H on the same vertex set, is there a graph K such that $G \subseteq K \subseteq H$ and K has an outside-obstacle representation. Our proof also shows that the *simple-polygon visibility graph sandwich problem*, the *inside-obstacle graph sandwich problem*, and the *single-obstacle graph sandwich problem* are all NP-hard.

1 Introduction

Recognizing graphs that have a certain type of geometric representation is a well-established field of research dealing with, for example, interval graphs, unit disk graphs, coin graphs (which are exactly the planar graphs), and visibility graphs. In this paper, we are interested in visibilities of points in the presence of a single obstacle. Given a (topologically open) non-self-intersecting polygon C and a finite set P of points in general position in the complement of C , the

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visibility graph $G_C(P)$ has a vertex for each point in P and an edge pq for any two points p and q in P that can *see* each other, that is, $\overline{pq} \cap C = \emptyset$. Given a graph G , we want to compute a (single-) obstacle representation of G , that is, an obstacle C and a set of points P such that $G = G_C(P)$ (if such a representation exists). The complexity of this reconstruction problem is open, even for the case that the points are exactly the vertices of a simple polygon and the (outside) obstacle is the complement of the polygon. This special case is called the *simple-polygon visibility graph (reconstruction) problem*.

The *visibility drawing* is a straight-line drawing of the visibility graph. The visibility drawing allows us to differentiate two types of obstacles: *outside* obstacles lie in the unbounded component of the visibility drawing, whereas *inside* obstacles lie in the complement of the unbounded component.

If we drop the restriction to single obstacles, our problem can be seen as an optimization problem. For a graph G , let $\text{obs}(G)$ be the smallest number of obstacles that suffices to represent G as a visibility graph. Analogously, let $\text{obs}_{\text{out}}(G)$ be the number of obstacles needed to represent G in the presence of an outside obstacle, and let $\text{obs}_{\text{in}}(G)$ be the number of obstacles needed to represent G in the absence of outside obstacles. Specifically, we say that G has an *outside-obstacle representation* if G can be represented by a single outside obstacle (e.g. Fig. 1), and G has an *inside-obstacle representation* if G can be represented by a single inside obstacle (e.g. Fig. 3b).

Previous work. Not only have Alpert et al. [1] introduced the notion of the obstacle number of a graph, they also characterized the class of graphs that can be represented by a single simple obstacle, namely a convex polygon. They also asked many interesting questions, for example, given an integer o , is there a graph of obstacle number exactly o ? If the previous question is true, given an integer $o > 1$, what is the smallest number of vertices of a graph with obstacle number o ? Mukkamala et al. [12] showed the first question is true. For the second question, Alpert et al. [1] found a 12-vertex graph that needs two obstacles, namely $K_{5,7}^*$, where $K_{m,n}^*$ with $m \leq n$ is the complete bipartite graph minus a matching of size m . They also showed that for any $m \leq n$, $\text{obs}(K_{m,n}^*) \leq 2$. This result was improved by Pach and Sariöz [13] who showed that the 10-vertex graph $K_{5,5}^*$ also needs two obstacles. More recently, Berman et al. [3] suggested some necessary conditions for a graph to have obstacle number 1 which they used to find a *planar* 10-vertex graph that cannot be represented by a single obstacle.

Alpert et al. [1] conjectured that every graph of obstacle number 1 has also outside-obstacle number 1. Berman et al. [3] further conjectured that every graph of obstacle number o has outside-obstacle number o . Alpert et al. [1] also showed that outerplanar graphs always have outside-obstacle representations and posed the question to bound the inside/convex obstacle number of outerplanar/planar graphs. Fulek et al. [6] partly answered this by showing that five convex obstacles are sufficient for outerplanar graphs—and that sometimes four are needed.

For the asymptotic bound on the obstacle number of a graph, it is obvious that any n -vertex graph has obstacle number $O(n^2)$. Balko et al. [2] showed that the obstacle number of an n -vertex graph is (at most) $O(n \log n)$. For the lower

bound, improving on previous results [1,12,11], Dujmović and Morin [5] showed there are n -vertex graphs whose obstacle number is $\Omega(n/(\log \log n)^2)$.

Johnson and Sariöz [9] investigated the special case where the visibility graph is required to be plane. They showed (by reduction from PLANARVERTEXCOVER) that in this case computing the obstacle number is NP-hard. By reduction to MAXDEG-3 PLANARVERTEXCOVER, they showed that the problem admits a polynomial-time approximation scheme and is fixed-parameter tractable. Koch et al. [10] also considered the plane case, restricted to outside obstacles. They gave a(n efficiently checkable) characterization of all biconnected graphs that admit a plane outside-obstacle representation.

A few years ago, Ghosh and Goswami [7] surveyed visibility graph problems, among them simple-polygon visibility graph problem. Open Problem 29 in their survey is the complexity of the recognition problem and Open Problem 33 is the complexity of the fore-mentioned reconstruction problem. Very recently, this question has been settled for an interesting variant of the problem where the points are not only the vertices of the graph but also the obstacles (which are closed in this case): Cardinal and Hoffmann [4] showed that recognizing point-visibility graphs is $\exists\mathbb{R}$ -complete, that is, as hard as deciding the existence of a real solution to a system of polynomial inequalities (and hence, at least NP-hard).

The graph sandwich problem has been introduced by Golumbic et al. [8] as a generalization of the recognition problem. They set up the abstract problem formulation and gave efficient algorithms for some concrete graph properties—and hardness results for others.

Preliminaries. In this paper, we consider only finite simple graphs. Whenever we say cycles, we always mean simple cycles. Let G be a graph and let v, u be its vertices. The *circumference* of G , denoted by $\text{circ}(G)$, is the length of its longest cycle. $v \sim u$ denotes that v and u are adjacent. We call v and u *twins* if $v \neq u$ and $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. We say v is *exposed to the outside* if it is on the boundary of the unbounded component of the straight-line drawing of G given by the point set. All vertices are exposed to the outside in an *exposed outside-obstacle representation*. In all figures (of graphs), unless otherwise stated, edges are solid and non-edges are dashed.

Our contribution. We have the following results. (Recall that a *co-bipartite* graph is the complement of a bipartite graph.)

- Every graph of circumference at most 6 has an outside-obstacle representation (Theorem 1).
- Every 7-vertex graph has an outside-obstacle representation (Theorem 2). Moreover, there is an 8-vertex co-bipartite graph that has no single-obstacle representation (Theorem 5).
- There is an 11-vertex co-bipartite graph with an inside-obstacle representation, but no outside-obstacle representation (Theorem 4). This resolves the above-mentioned open problems of Alpert et al. [1] and Berman et al. [3].
- The Outside-Obstacle Graph Sandwich Problem is NP-hard even for co-bipartite graphs. The same holds for the Simple-Polygon Visibility Graph Sandwich Problem. This does not solve, but sheds some light on a long-standing open problem: the recognition of visibility graphs of simple poly-

gons. While little is known for the complexity of computing the obstacle number, the Single-Obstacle Graph Sandwich Problem is shown to be also NP-hard.

Remarks and Open Problems. The recognition of inside- and outside-obstacle graphs is currently open. We expect that testing either of these cases is NP-hard. Assuming that this is true, it would be interesting to show fixed-parameter tractability w.r.t. the number of vertices of the obstacle. We now know that $\text{obs}_{\text{in}}(G)$ and $\text{obs}_{\text{out}}(G)$ are usually different, but can we bound $\text{obs}_{\text{in}}(G)$ in terms of $\text{obs}_{\text{out}}(G)$? While we have shown that the trivial lower bound $\text{obs}_{\text{out}}(G) - 1$ is tight, an upper bound is only known for outerplanar graphs [1,6].

2 Graphs with Small Circumference

In this section we will describe how to construct an outside-obstacle representation for any graph whose circumference is at most 6. To prove this result we show that for every vertex v of a biconnected graph G with circumference at most 6, there is an exposed outside-obstacle representation of G with v on the convex hull of $V(G)$. Lemma 3 makes it easier to describe the outside-obstacle representation. We then apply Lemma 1 and Lemma 2 to obtain an outside-obstacle representation of a graph.

We provide an 8-vertex graph of circumference 8 that requires at least two obstacles in the next section, so the only gap is the circumference-7 case. We conjecture that every graph of circumference 7 has an outside-obstacle representation. As a first step towards this conjecture, we show that every 7-vertex graph has an outside-obstacle representation by providing a list of point sets such that each 7-vertex graph can be represented by an outside obstacle when the vertices of the graph are mapped to a point set in our list.

Proofs of Lemmas 1,2,3 are in Appendix A and brief ideas are sketched here.

Lemma 1 *Let G and H be graphs on different vertex sets. If $\text{obs}_{\text{out}}(G) = 1$ and $\text{obs}_{\text{out}}(H) = 1$, then $\text{obs}_{\text{out}}(G \cup H) = 1$.*

Proof (sketch). Place two graphs far enough and merge outside obstacles. \square

Lemma 2 *Let G and H be graphs with exposed outside-obstacle representations. Let u be a vertex of G , and let v be a vertex of H . Assume that v lies on the convex hull of $V(H)$. If K is the graph obtained by identifying u and v , then K also has an exposed outside-obstacle representation.*

Proof (sketch). Make the outside-obstacle representation of H small and narrow (with respect to v) enough to fit in some circular sector lying inside the obstacle centered at u in the outside-obstacle representation of G . Then replace the circular sector with above obstacle representation of H . \square

Lemma 3 *Let H be a graph, v be a vertex of H , A be the set of twins of v , and $G = H \setminus A$. If G has an exposed outside-obstacle representation in which v lies on the convex hull of $V(G)$, then H has an exposed outside-obstacle representation in which all vertices in $A \cup \{v\}$ lie on the convex hull of $V(H)$.*

Proof (sketch). Place twins close enough since their neighborhoods are same. \square

The following observation helps to restrict the structure of biconnected graphs of given circumference where indices are taken modulo k .

Observation 1 *Let G be a graph of circumference k and let $C = v_1v_2 \dots v_k$ be a cycle. G doesn't contain a $v_i - v_{i+t}$ path P of length t' disjoint to v_iCv_{i+t} where $0 < t < k$ and $t' > t$, since it would create $(k + t' - t)$ -cycle. In particular, if $v \notin \{v_1, \dots, v_k\}$ is adjacent to v_i , then v is neither adjacent to v_{i-1} nor v_{i+1} .*

Theorem 1 *If the circumference of a graph G is at most 6, then G has an outside-obstacle representation.*

Proof. If G is disconnected, we give an outside-obstacle representation for each connected component and simply merge them by Lemma 1.

When G is connected, we decompose it into its biconnected components, i.e., the block decomposition tree of G . Starting in its root, we include representations of the children in turn using Lemma 2.

Let H be a biconnected component of G . It suffices to show that H satisfies the condition for Lemma 2: For each vertex v of H , H has an exposed outside-obstacle representation such that v is on the convex hull of $V(H)$.

Case 1: $\text{circ}(H) = 3$

As H is biconnected, H is a triangle and trivially satisfies the condition.

Case 2: $\text{circ}(H) = 4$

Let $C = v_1v_2v_3v_4 \subset H$ be a 4-cycle. If H contains exactly four vertices, there is an outside-obstacle representation; see Fig. 1a. Note that we can choose the (dashed blue) diagonals v_1v_3 and v_2v_4 to be edges or non-edges as desired. Otherwise, without loss of generality, there is a vertex $x \in H \setminus C$ with $x \sim v_1$. As H is biconnected, there is a path of length at least 2 from v_1 to another vertex of C containing x . Observation 1 implies that $x \not\sim v_2$, $x \sim v_3$, and $x \not\sim v_4$. Since we have another 4-cycle $C' = v_1xv_3v_4$, the same holds for v_2 , implying $v_2 \not\sim v_4$. Hence x is a non-adjacent twin of v_2 . It follows that any vertex in $H \setminus C$ is a non-adjacent twin of one of v_1, \dots, v_4 . Since the vertices in Fig. 1a are in convex position, we can embed H using Lemma 3.

Case 3: $\text{circ}(H) = 5$

Let $C = v_1v_2v_3v_4v_5 \subset H$ be a 5-cycle. If H contains exactly five vertices, see Fig. 1b for its outside-obstacle representation. Otherwise, without loss of generality, there is a vertex $x \in H \setminus C$ with $x \sim v_1$. Observation 1 implies $x \not\sim v_2, v_5$. As H is biconnected, there is either path v_1xv_3 or v_1xv_4 . Without loss of generality, we assume $x \sim v_3$ and thus $x \not\sim v_4$. Then $v_2 \not\sim v_4, v_5$ since we have another 5-cycle $v_1xv_3v_4v_5$ and can apply the same logic. Hence, x is a non-adjacent twin of v_2 . As in the Case 2, we see that every vertex in $H \setminus C$ is a non-adjacent twin of one of v_1, v_2, \dots, v_5 and we can embed H using Lemma 3.

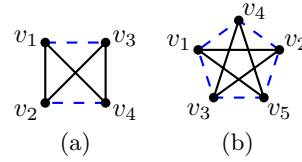


Fig. 1: Graphs of circumference 4 and 5 with outside-obstacle representations

Case 4: $\text{circ}(H) = 6$ (We postpone this case to Appendix A.) \square

Theorem 2 *Any graph with at most 7 vertices has an outside-obstacle representation.*

Proof (sketch). By Theorem 1, it suffices to provide an outside-obstacle representation of each 7-vertex graph containing C_7 . In Appendix A, we classify such graphs into 15 groups and give an outside-obstacle representation of each. \square

3 Co-Bipartite Graphs

We now consider obstacle representations of *co-bipartite* graphs. Recall that a graph is co-bipartite if its complement is bipartite. Using this seemingly simple graph class, we settle an open problem posed by Alpert et al. [1] who asked if each graph with obstacle number 1 has an outside-obstacle representation. Namely, we provide an 11-vertex graph B_{11} (see Fig. 3b) where not only is this not the case, but B_{11} in fact has an inside-obstacle representation where the obstacle is the simplest possible shape, i.e., a triangle.³ We also provide a smallest graph with obstacle number 2; see the 8-vertex graph in Fig. 3c. This improves on the smallest previously known such graphs (e.g., the 10-vertex graphs of Pach and Sariöz [13] and of Berman et al. [3]) and shows that Theorem 2 is tight.

Properties of Outside-Obstacle Representations. We build on the easy observation (see Observation 2 below) that in every outside-obstacle representation of a graph, for every clique Z , the convex hull $\text{CH}(Z)$ of the point set of Z cannot be touched by the obstacle. In other words, the obstacle must occur outside of each such convex hull. Since we focus on co-bipartite graphs, this observation greatly restricts the ways one may realize an outside representation. Additionally, we will use this observation implicitly throughout this section whenever considering two cliques in a graph with an outside-obstacle representation.

Observation 2 *If G has an outside-obstacle representation (P, C) , then for every clique $Z \subseteq V(G)$, the convex hull $\text{CH}(Z)$ of the points corresponding to Z is disjoint from C , i.e., $C \cap \text{CH}(Z) = \emptyset$.*

For a graph G containing two cliques $Z, Z' \subseteq V(G)$ and outside-obstacle representation, consider the convex hulls $\text{CH}(Z)$ and $\text{CH}(Z')$. We say that these convex hulls are *k-crossing* when $\text{CH}(Z) \setminus \text{CH}(Z')$ consists of $k+1$ disjoint regions. Note that this condition is symmetric, i.e., when $\text{CH}(Z) \setminus \text{CH}(Z')$ consists of r disjoint regions so does $\text{CH}(Z') \setminus \text{CH}(Z)$. We refer to these disjoint regions of the difference as the *petals* of Z (Z' respectively).

We now introduce a special 6-vertex graph K_6^* which is used in the following technical lemma and our NP-hardness proof. This graph is the result of deleting a 3-edge matching from a 6-clique; see Fig. 3a.

³ Note that for topologically closed obstacles, this obstacle could be a line segment.

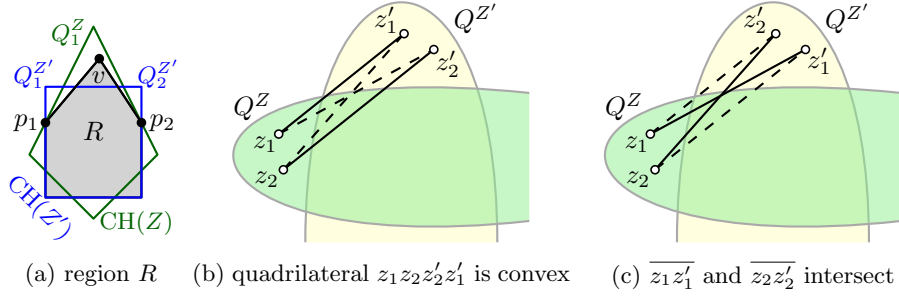


Fig. 2: Aides for the proof of Lemma 4.

Lemma 4 *Let G be a graph containing two cliques Z, Z' . For every outside-obstacle representation of G , the following properties hold.*

- (a) *If $\text{CH}(Z)$ and $\text{CH}(Z')$ are t -crossing, then every vertex in Z has at least $t - 1$ neighbors in Z' and vice versa. That is, if Z contains a vertex with only r neighbors in Z' , then $\text{CH}(Z)$ and $\text{CH}(Z')$ are at most $(r + 1)$ -crossing.*
- (b) *If G contains K_6^* (with missing edges $z_1z_1', z_2z_2', z_3z_3'$; see Fig. 3a) as an induced subgraph, $\{z_1, z_2, z_3\} \subseteq Z$, and $\{z_1', z_2', z_3'\} \subseteq Z'$, then $\text{CH}(\{z_1, z_2, z_3\})$ and $\text{CH}(\{z_1', z_2', z_3'\})$ are at least 1-crossing. Furthermore, $\text{CH}(Z)$ and $\text{CH}(Z')$ are at least 1-crossing.*
- (c) *If G contains a 4-cycle $z_1z_1'z_2'z_2$ as an induced subgraph, $\{z_1, z_2\} \subseteq Z$, $\{z_1', z_2'\} \subseteq Z'$, $\text{CH}(Z)$ and $\text{CH}(Z')$ intersect, and z_1 and z_2 are contained in a petal Q^Z of Z , then z_1' and z_2' are contained in different petals of Z' which are both adjacent to Q^Z . This implies that, if $\text{CH}(Z)$ and $\text{CH}(Z')$ are 1-crossing, then either z_1 and z_2 or z_1' and z_2' are in different petals.*

Proof. (a) Suppose $\text{CH}(Z)$ and $\text{CH}(Z')$ are t -crossing for some $t \geq 2$. Note that $|Z|, |Z'| \geq t + 1$ since the convex hull of each must contain at least $t + 1$ points. For $A \in \{Z, Z'\}$, let Q_0^A, \dots, Q_t^A be the petals of $\text{CH}(A)$ in clockwise order around $\text{CH}(Z) \cap \text{CH}(Z')$ where, for each $i \in \{0, \dots, t\}$, Q_i^Z is between $Q_i^{Z'}$ and $Q_{i+1}^{Z'}$ and all indices are considered modulo $t + 1$.

Consider a vertex $v \in Z$ ($v \in Z'$ follows symmetrically). If v is in $\text{CH}(Z) \cap \text{CH}(Z')$, then we are done since v sees every vertex in Z' and $|Z'| \geq t + 1$. So, suppose $v \in Q_1^Z$. Consider the points $p_1 = Q_1^{Z'} \cap Q_0^Z$ and $p_2 = Q_2^{Z'} \cap Q_2^Z$. Define the subregion R (depicted as the grey region in Fig. 2a) of $\text{CH}(Z) \cup \text{CH}(Z')$ whose boundary, in clockwise order, is formed by $\overline{p_1v}$, $\overline{vp_2}$, and the polygonal chain from p_2 to p_1 along the boundary of $\text{CH}(Z')$. Note that, for each $i \in \{0, 3, 4, \dots, t\}$, $Q_i^{Z'} \subset R$ and R is convex, i.e., for every $u \in Q_i^{Z'}$, the line segment \overline{vu} is contained in $\text{CH}(Z) \cup \text{CH}(Z')$. Thus, v has at least $t - 1$ neighbors in Z' .

(b) Consider the graph K_6^* as labeled in Fig. 3a. We first show that the convex hulls of $X = \{z_1, z_2, z_3\}$ and $Y = \{z_1', z_2', z_3'\}$ are at least 1-crossing.

Suppose that $\text{CH}(X)$ and $\text{CH}(Y)$ intersect but are 0-crossing. Since $|X| = |Y| = 3$, a vertex in $X \cup Y$ must be contained in $\text{CH}(X) \cap \text{CH}(Y)$. Hence, this vertex dominates $X \cup Y$, but K_6^* doesn't have such a vertex—a contradiction.

Now, suppose that $\text{CH}(X)$ and $\text{CH}(Y)$ are disjoint, and let $H = \text{CH}(X \cup Y)$. Since $\text{CH}(X)$ and $\text{CH}(Y)$ are disjoint, the boundary ∂H of H contains at most two line segments that connect a vertex of X to a vertex of Y , i.e., at most two non-edges of K_6^* occur on ∂H . However, we will now see that every non-edge of K_6^* must occur on ∂H . Consider the line segment $\overline{z_1 z'_1}$ and suppose it is not on ∂H . This means that there are vertices u and v of $K_6^* \setminus \{z_1, z'_1\}$ where u and v occur on opposite sides of the line determined by $\overline{z_1 z'_1}$. However, since $\overline{z_1 z'_1}$ is the only non-edge incident to either z_1 or z'_1 , the non-edge $\overline{z_1 z'_1}$ is enclosed by $\overline{uz_1}$, $\overline{z_1 v}$, $\overline{vz'_1}$, $\overline{z'_1 u}$, which provides a contradiction. Thus, every non-edge must occur on ∂H , which contradicts the fact that at most two line segments spanning between $\text{CH}(X)$ and $\text{CH}(Y)$ can occur on ∂H .

We now know that $\text{CH}(X)$ and $\text{CH}(Y)$ are at least 1-crossing. We use this to observe that $\text{CH}(Z)$ and $\text{CH}(Z')$ must also be at least 1-crossing. Clearly, if $\text{CH}(Z)$ and $\text{CH}(Z')$ are disjoint, this contradicts $\text{CH}(X)$ and $\text{CH}(Y)$ being at least 1-crossing. So, suppose that $\text{CH}(Z)$ and $\text{CH}(Z')$ intersect but are not 1-crossing. Note that no vertex v of K_6^* is contained in $\text{CH}(Z) \cap \text{CH}(Z')$ since otherwise v would dominate to K_6^* . In particular, $X \subseteq \text{CH}(Z) \setminus \text{CH}(Z')$ and $Y \subseteq \text{CH}(Z') \setminus \text{CH}(Z)$. However, we again would have $\text{CH}(X)$ and $\text{CH}(Y)$ being disjoint, i.e., a contradiction. Thus, $\text{CH}(Z)$ and $\text{CH}(Z')$ are at least 1-crossing.

(c) Suppose that z'_1 and z'_2 belong to the same petal $Q^{Z'}$. This petal is adjacent to Q^Z , as otherwise z_1 would be visible to z'_2 (i.e., providing a contradiction). Now, if the quadrilateral $z_1 z_2 z'_2 z'_1$ is convex, the non-edge $\overline{z_1 z'_2}$ is not accessible from the outside (see Fig. 2b). If the quadrilateral $z_1 z_2 z'_2 z'_1$ is non-convex, either a non-edge $\overline{z_1 z'_2}$ or a non-edge $\overline{z_2 z'_1}$ will not be accessible from the outside. Thus, $\overline{z_1 z'_1}$ and $\overline{z_2 z'_2}$ intersect since $\text{CH}(\{z_1, z_2\})$ and $\text{CH}(\{z'_1, z'_2\})$ are disjoint. The edge $\overline{z_1 z'_1}$ together with the boundary of $\text{CH}(Z) \cup \text{CH}(Z')$ split the plane into at most two bounded and one unbounded region. Then at least one of the non-edges $\overline{z_1 z'_2}$ and $\overline{z'_1 z_2}$ lies inside the union of the bounded regions. This contradicts the fact that all non-edges should be accessible from the outside. For example, in Fig. 2c, the non-edge $\overline{z'_1 z_2}$ cannot intersect any outside obstacle. \square

Inside- vs. Outside-Obstacle Graphs. We now use Lemma 4 to show that there is an 11-vertex graph (see B_{11} in Fig. 3b) that has an inside-obstacle representation but no outside-obstacle representation. This resolves an open question of Alpert et al. [1]. We conjecture that, for any graph G with at most 10 vertices, $\text{obs}_{\text{in}}(G) = 1$ implies $\text{obs}_{\text{out}}(G) = 1$.

Theorem 3 *There is an 11-vertex graph (e.g., B_{11} in Fig. 3b)*

Proof. The 11-vertex co-bipartite graph B_{11} is constructed as follows. We start with K_{10} on the vertices $z_1, \dots, z_5, z'_1, \dots, z'_5$. We then delete a 5-edge matching $\{\overline{z_i z'_i} : i \in \{1, \dots, 5\}\}$ from K_{10} to obtain K_{10}^* . Finally, we obtain B_{11} by adding a vertex v adjacent to z_1, \dots, z_5 . (Fig. 3b shows an inside-obstacle representation of B_{11} with a triangular obstacle.)

It remains to argue that B_{11} has no outside-obstacle representation. Note that B_{11} contains two cliques $Z = \{z_1, \dots, z_5, v\}$ and $Z' = \{z'_1, \dots, z'_5\}$. Furthermore,

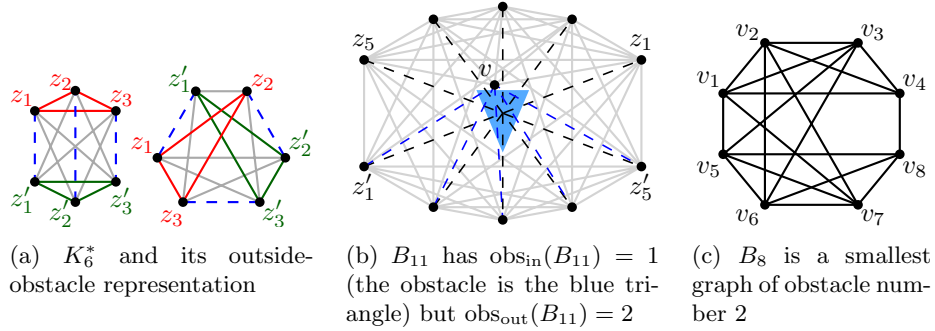


Fig. 3: Three small graphs: K_6^* , B_{11} and B_8

the vertex $v \in Z$ has no neighbors in Z' . Thus, by Lemma 4 (a), in any outside-obstacle representation, $\text{CH}(Z)$ and $\text{CH}(Z')$ are at most 1-crossing. Additionally, since each z_i has a non-neighbor in Z' , no z_i is contained in $\text{CH}(Z) \cap \text{CH}(Z')$. In particular, since Z has only two petals, there are three z_i 's, say z_1, z_2, z_3 , that are contained in a single petal of Z . Now note that K_6^* is the subgraph of B_{11} induced by $\{z_1, z_2, z_3, z_1', z_2', z_3'\}$. Since z_1, z_2, z_3 are contained in a petal of Z , $\text{CH}(\{z_1, z_2, z_3\})$ and $\text{CH}(\{z_1', z_2', z_3'\})$ are disjoint, contradicting Lemma 4 (b). Thus, B_{11} has outside-obstacle number 2. \square

Note that a graph with an inside-obstacle representation is either a clique or contains a cycle since an inside obstacle cannot (by definition) pierce the convex hull of the point set⁴. Thus, by Theorem 3 and this fact, we have the following.

Theorem 4 *The classes of inside-obstacle representable graphs and outside-obstacle representable graphs are incomparable.*

Obstacle Number 2. We present an 8-vertex graph (see B_8 in Fig. 3c) with obstacle number 2. To prove this result, we first apply Lemma 4 to show that B_8 has no outside-obstacle representation. In Lemma 5 (proven in Appendix B), we demonstrate that B_8 also has no inside-obstacle representation. In particular, these lemmas together with Theorem 2 provide the following theorem.

Theorem 5 *The smallest graphs without a single-obstacle representation have eight vertices, e.g., the co-bipartite graph B_8 in Fig. 3c.*

Proof. The graph B_8 has 8 vertices v_1, \dots, v_8 . It has precisely the following set of non-edges: $v_1v_6, v_2v_5, v_3v_7, v_4v_5, v_4v_6, v_4v_7, v_8v_1, v_8v_2, v_8v_3$. Note that the subgraph induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$ is a K_6^* . Further, note that $Z = \{v_1, v_2, v_3, v_4\}$ and $Z' = \{v_5, v_6, v_7, v_8\}$ are cliques.

Suppose (for a contradiction) B_8 has an outside-obstacle representation. By Lemma 4 (b), $\text{CH}(Z)$ and $\text{CH}(Z')$ are at least 1-crossing. Additionally, since

⁴ In Appendix D, we show that $K_{2,3}$ is the smallest graph with a cycle and an outside-obstacle representation but no inside-obstacle representation.

v_4 has only one neighbor in Z' , we know that $\text{CH}(Z)$ and $\text{CH}(Z')$ are at most 2-crossing. We will consider these two cases separately. Let Q_0^Z, Q_1^Z, Q_2^Z be the petals of Z and $Q_0^{Z'}, Q_1^{Z'}, Q_2^{Z'}$ be the petals of Z' where the cyclic order of the petals around $\text{CH}(Z) \cap \text{CH}(Z')$ is $Q_0^{Z'}, Q_0^Z, Q_1^{Z'}, Q_1^Z, Q_2^{Z'}, Q_2^Z$. Note that every vertex is contained in one of the petals.

Case 1: $\text{CH}(Z)$ and $\text{CH}(Z')$ are 2-crossing. Suppose $v_4 \in Q_0^Z$. Since v_8 is the only neighbor of v_4 in Z' , we must have $v_8 \in Q_2^{Z'}$, and now the only vertex in Q_0^Z is v_4 and the only vertex in $Q_2^{Z'}$ is v_8 . However, we now have $\{v_1, v_2, v_3\} \subset Q_1^Z \cup Q_2^Z$ and $\{v_5, v_6, v_7\} \subset Q_0^{Z'} \cup Q_1^{Z'}$, i.e., $\text{CH}(\{v_1, v_2, v_3\})$ and $\text{CH}(\{v_5, v_6, v_7\})$ are disjoint, contradicting Lemma 4 (b).

Case 2: $\text{CH}(Z)$ and $\text{CH}(Z')$ are 1-crossing. Note that v_1, v_2 , and v_3 cannot belong to the same petal (otherwise, we would contradict Lemma 4 (b)). Similarly, v_5, v_6 , and v_7 cannot belong to the same petal. Thus, without loss of generality, we have v_1 and v_2 in Q_0^Z , v_3 in Q_1^Z , v_5 and v_7 in $Q_0^{Z'}$, and v_6 in $Q_1^{Z'}$. When v_4 is in Q_0^Z and v_8 is in $Q_2^{Z'}$, the induced 4-cycle $v_4v_2v_7v_8$ contradicts Lemma 4 (c). Similarly, when v_4 is in Q_0^Z and v_8 is in $Q_1^{Z'}$, we use the induced 4-cycle $v_4v_2v_6v_8$; when v_4 is in Q_1^Z and v_8 is in $Q_0^{Z'}$, we use the induced 4-cycle $v_4v_3v_5v_8$; and when v_4 is in Q_1^Z and v_8 is in $Q_1^{Z'}$, we use the induced 4-cycle $v_4v_3v_6v_8$.

It remains to show that B_8 has no inside-obstacle representation (formalized in Lemma 5 below). This is proven in Appendix B. \square

Lemma 5 *The graph B_8 in Fig. 3c has no inside-obstacle representation.*

4 NP-Hardness

In this section, we show that the single-obstacle, outside-obstacle, inside-obstacle graph sandwich problems as well as the simple-polygon visibility graph sandwich problem are all NP-hard. Note that the complexity of the obstacle graph sandwich problem yields an upper bound for the complexity of our (simpler) recognition problem.

Theorem 6 *The outside-obstacle graph sandwich problem is NP-hard. In other words, given two graphs G and H with the same vertex set and $G \subseteq H$, it is NP-hard to decide whether there is a graph K such that $G \subseteq K \subseteq H$ and $\text{obs}_{\text{out}}(K) = 1$. This holds even if G and H are co-bipartite.*

Proof. We reduce from MONOTONE NOT ALL EQUAL 3SAT, which is NP-hard [14]. In this version of 3SAT, all literals are positive, and the task is to decide whether the given 3SAT formula φ admits a truth assignment such that in each clause at least one and at most two variables are true.

Given φ , we build a graph G_φ with edges, non-edges and “maybe”-edges such that φ is a yes-instance if and only if G_φ has a subgraph that has an outside-obstacle representation and contains all edges, no non-edges and an arbitrary subset of the maybe-edges. (In other words, the set of edges of G_φ yields G in the statement of the theorem, and the set of edges and maybe-edges yields H .)

Let $\{v_1, \dots, v_n\}$ be the set of variables, and let $\{C_1, \dots, C_m\}$ be the set of clauses in φ . For $i = 1, \dots, n$, let v_{ij} be the j -th occurrence of v_i in φ .

Now we can construct G_φ . For each variable, we introduce a *variable vertex* (of the same name). These n vertices form a clique. For each occurrence v_{ij} of a variable v_i in φ , we introduce an *occurrence vertex* (of the same name). These $3m$ vertices also form a clique. In order to restrict how the two cliques intersect, we add to G_φ a copy of K_6^* labeled as in Fig. 3a; vertices z_1, z_2, z_3 participate in the occurrence-vertex clique, whereas vertices z'_1, z'_2, z'_3 participate in the variable-vertex clique. We add one more vertex u to the occurrence-vertex clique. The special vertex u is adjacent to z'_3 and has non-edges to all other vertices in the variable-vertex clique. The edge set of G_φ depends on φ as follows. Each variable vertex v_i has

- an edge to any occurrence vertex v_{ij} ,
- a non-edge to any occurrence vertex $v_{k\ell}$ that represents an occurrence of a variable v_k that co-occurs with v_i in some clause of φ ,
- a maybe-edge to any other occurrence vertex.

Next, we show how to use a feasible truth assignment of φ to lay out G_φ so that all its non-edges are accessible from the outside. We place the vertices on the boundary of two intersecting rectangles, one for each clique. Given these positions, we show that all non-edges intersect the outer face of the union of the edges. Finally, we bend the sides of the rectangles slightly into very flat circular arcs such that all of the previous (non-) visibilities remain and the vertices are in general position.

We take two axis-aligned rectangles R_1 and R_2 that intersect as a cross; see Fig. 4. Let X_1, X_2, X_3, X_4 be the corners of $R_1 \cap R_2$ in clockwise order, starting in the lower left corner. We place the variable vertices on the boundary of the “wide” rectangle R_1 : the vertices v_1, \dots, v_p of the true variables are equally spaced from top to bottom on a segment on the left side, similarly the vertices v_{p+1}, \dots, v_n of the false variables go to a segment on the right side. (In Fig. 4(b), $p = 3$.) The two vertical segments are chosen such that they “see” four disjoint horizontal segments on the top and bottom edge of R_2 ; refer to Fig. 4(a) for the positions of the six segments in total.

In each clause, we sort the variables in increasing order of index. We place the occurrence vertices on the horizontal segments of R_2 . For a true variable v_i (such as v_2 in Fig. 4(b)) the first occurrence vertex v_{i1} has two potential locations; the bottom location is where the ray from v_i through X_1 hits the bottom right segment, the top location is where the ray from v_i through X_2 hits the top right segment. We place v_{i1} to its bottom or top location depending on whether v_{i1} is the first or second occurrence of a true variable in its clause, respectively. (Remember that within each clause, at most two variables are true and at most two are false.) Occurrence vertices v_{i2} etc. go between the top or bottom locations of v_{i1} and $v_{i+1,1}$, again depending on whether they are the first or second occurrence of a true variable in their respective clauses. (E.g., in Fig. 4(b)), v_{21} goes to the top, whereas v_{22} goes to the bottom.)

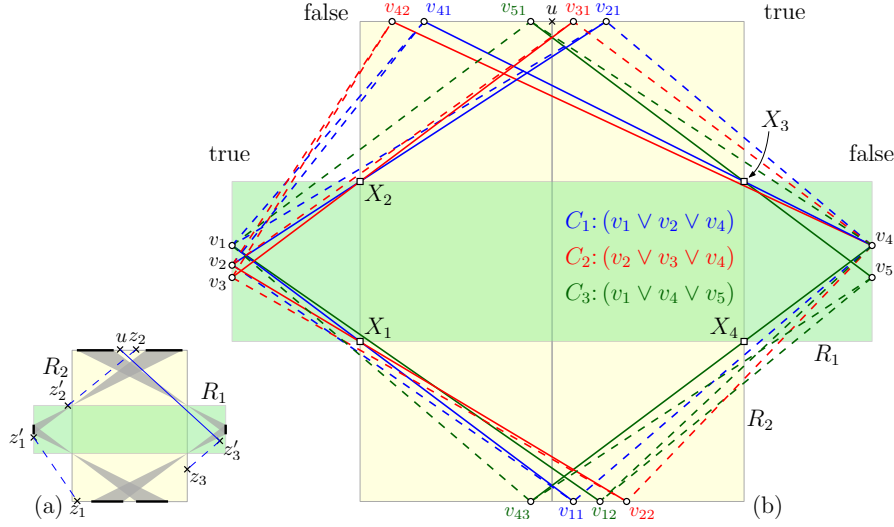


Fig. 4: NP-hardness: maybe-edges and the two cliques are not drawn.

The special vertex u is placed in the center of the top edge of R_2 ; hence, it is not visible from any variable vertex; see Fig. 4(a). The vertices of K_6^* can be placed such that u sees only z'_3 , but neither z'_1 nor z'_2 ; see Fig. 4(b)).

By construction, all edges are inside $R_1 \cup R_2$. It remains to show that all non-edges (dashed in Fig. 4(b)) go through the complement of $R_1 \cup R_2$. This is due to the order of the variable vertices and the occurrence vertices along the boundary of $R_1 \cup R_2$ and due to the order of the variables in each clause. Suppose that a variable vertex v_i has a non-edge with occurrence vertex $v_{k\ell}$. This means that there is an occurrence v_{ij} of v_i in the same clause as $v_{k\ell}$. If v_i and v_k have different truth values, then v_i cannot see $v_{k\ell}$; refer to Fig. 4(a). So assume that both are true and that $i < k$. But then v_i lies above v_k on the left segment of R_1 , and v_{ij} lies to the left of $v_{k\ell}$ on the bottom right segment of R_2 . Hence, v_i cannot see $v_{k\ell}$.

It remains to show that an outside-obstacle representation of G_φ yields a feasible truth assignment for φ . By Lemmas 4(a) and (b), we know that the convex hulls of the two cliques are at least 1-crossing due to the presence of K_6^* and at most 2-crossing due to u . To see that these hulls are exactly 1-crossing, we suppose that G_φ has a 2-crossing drawing for a contradiction. Consider the subgraph H induced by u and the first clause $C_1 = \{v_i, v_j, v_k\}$, of φ i.e., $H = G[\{u, v_i, v_j, v_k, v_{i1}, v_{j1}, v_{k1}\}]$. Let Q_u be the petal containing u . Since the only neighbor of u in the variable-vertex clique is z'_3 , no other variable vertices belong to the petal opposite Q_u . Thus, two of $\{v_i, v_j, v_k\}$, say v_i and v_j , occur in one petal Q'_1 adjacent to Q_u , and v_k occurs in the other petal Q'_2 which is adjacent to Q_u . Notice that each of v_{i1}, v_{j1}, v_{k1} cannot belong to the petal opposite Q'_1 since this would make it adjacent to both v_i and v_j . Similarly, no neither v_{i1} nor v_{j1} can occur in the petal opposite Q'_2 since it would then be adjacent to v_k .

Thus, v_{i1} and v_{j1} belong to the same petal and this petal is adjacent to Q'_1 . However, this contradicts Lemma 4(c) since $\{v_i, v_j, v_{i1}, v_{j1}\}$ induces a 4-cycle.

Now, since the convex hulls are exactly 1-crossing, we have two groups (petals) of vertices in each of the two cliques. Without loss of generality, the variable-vertex clique is divided into a left and a right group, and the occurrence-vertex clique is divided into a top and a bottom group. We set those variables to true whose vertices lie on the left, the rest to false.

Now suppose that the three variables v_1 , v_2 , and v_3 of clause C_1 lie in the same group, say, on the left. Then two of their occurrence vertices (say v_{11} and v_{21}) lie in the same group, say, in the top group. Since v_1v_{21} and v_2v_{11} are non-edges, $v_1v_{11}v_{21}v_2$ is an induced 4-cycle. Now Lemma 4(c), yields the desired contradiction. Hence, no three variable vertices in a clause can be in the same (left or right) group. Therefore, our truth assignment is indeed feasible. This completes the NP-hardness proof. \square

To show hardness for the simple-polygon visibility graph sandwich problem, we must make sure that any vertex of the obstacle is also a vertex of the graph. It suffices to add X_1, X_2, X_3, X_4 as vertices to G_φ that lie in both cliques.

Theorem 7 *The simple-polygon visibility graph sandwich problem is NP-hard. In other words, given two graphs G and H with the same vertex set and $G \subseteq H$, it is NP-hard to decide whether there is a graph K and a polygon Π such that $G \subseteq K \subseteq H$ and $K = G_\Pi(V(\Pi))$. This holds even if G and H are co-bipartite.*

We can also use the NP-hardness of the outside-obstacle sandwich problem to show NP-hardness for both the single-obstacle sandwich problem and the inside-obstacle sandwich problem. The idea is simply to combine a given graph G with a graph such as B_{11} which has outside-obstacle number greater than one, but inside-obstacle number one. The combined graph would then have inside-obstacle number one if and only if the graph G has outside-obstacle number one. The details of this are given in Appendix C.

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Appendix

A Missing Proofs of Section 2

This appendix contains the omitted proof of lemmas for Theorem 1 and the missing part of proof for Theorem 1 and Theorem 2.

Lemma 1 *Let G and H be graphs on different vertex sets. If $\text{obs}_{\text{out}}(G) = 1$ and $\text{obs}_{\text{out}}(H) = 1$, then $\text{obs}_{\text{out}}(G \cup H) = 1$.*

Proof. Fix outside-obstacle representations for G and H . We can assume G lies inside $(0, 1) \times (0, 1)$ and H lies inside $(1, 2) \times (0, 1)$ by scaling. Let C_G be an outside obstacle for G and C_H for H . We can also assume that $\partial([0, 1] \times [0, 1]) \subset \partial C_G$ and $\partial([1, 2] \times [0, 1]) \subset \partial C_H$. Take $C = C_G \cup C_H \cup (\{1\} \times [0, 1])$. We claim that C is an obstacle for $G \cup H$. Let v be a vertex of G and u a vertex of H . Since \overline{vu} intersects with $\{1\} \times [0, 1]$ and $\{1\} \times [0, 1] \subset C$, we indeed have uv as a non-edge. See Fig. 5 for an example illustration. \square

Lemma 2 *Let G and H be graphs with exposed outside-obstacle representations. Let u be a vertex of G , and let v be a vertex of H . Assume that v lies on the convex hull of $V(H)$. If K is the graph obtained by identifying u and v , then K also has an exposed outside-obstacle representation.*

Proof. Fix an exposed outside-obstacle representation of H such that v is on the convex hull of $V(H)$. Let $C = z_1 z_2 \dots z_m z_1$ be the one of boundaries of the outside obstacle such that the obstacle lies in the unbounded component of $\mathbb{R}^2 \setminus C$ and all vertices of H lies in the bounded component. Let l, m be two rays starting in v where all vertices of H are between l and m . Since v is on the convex hull of $V(H)$, the angle between l and m is less than π . Without loss of generality, v is placed at the origin and the ray from v to $(1, 0)$ is between l and m .

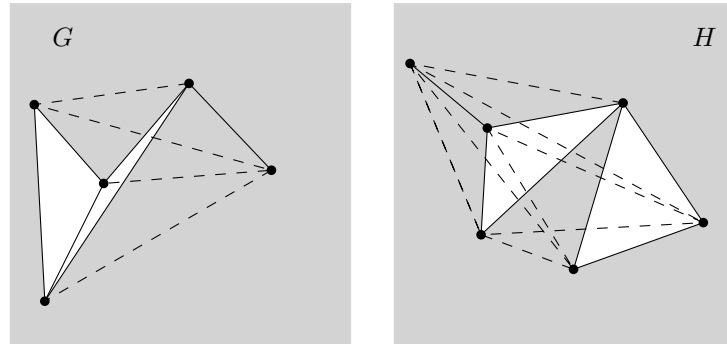


Fig. 5: Grey (open) regions are outside obstacles of G and H respectively

We first show that squashing and shrinking with respect to v preserves the structure of outside-obstacle representation. More precisely, for $s, t > 0$, let $T_{s,t}$ be a transformation mapping a point (x, y) to (sx, ty) . We show that the point set with the obstacle obtained by transforming each vertex of H and its outside obstacle by $T_{s,t}$ is still an exposed outside-obstacle representation of H where v is on the convex hull of $V(H)$. Let $C' = z'_1 \dots z'_m z'_1 = T_{s,t}(z_1) \dots T_{s,t}(z_m) T_{s,t}(z_1)$ and $a' = T_{s,t}(a)$ for $a \in V(H)$ and for simplicity. Let a, b be vertices of H . If $a \sim b$, suppose $\overline{a'b'}$ intersects with the obstacle for contradiction. It is clear that if $\overline{a'b'}$ intersects with $\overline{z'_i z'_{i+1}}$ then \overline{ab} intersects with $\overline{z_i z_{i+1}}$, contradicting the fact that ab should not intersect C . Similarly, we can also show that if $a \not\sim b$ then $\overline{a'b'}$ intersects with C' , that every transformed vertex is exposed to the outside, and that v' is on the convex hull of the transformed point set.

Consequently, we can assume that H has an exposed outside-obstacle representation where all vertices are contained in an open circular sector centered at v whose radius and angle are arbitrarily small such that the boundary of the outside obstacle includes the boundary of the arc. Fix an exposed outside-obstacle representation of G such that the outside obstacle is maximal (i.e., the outside obstacle is the unbounded component of the complement of the visibility drawing). Since u is exposed to the outside, we can find an arc sector A of radius r and angle θ centered at u , completely lying inside the outside obstacle except u . We replace A with above obstacle representation of H while identifying u and v . We claim that it is an outside-obstacle representation of K . Since all edges/non-edges of H lie inside A , they are properly represented with the new obstacle. Since new obstacle is a subset of obstacle of G , all edges of G don't intersect with the obstacle. For non-edges of G , if they didn't intersect A , they would still intersect the obstacle. Otherwise, since ∂A is contained in the obstacle, they would also intersect the obstacle. For a vertex of G except u and a vertex of H except v , they are always non-adjacent and it is properly represented since the line segment connecting them intersects with ∂A . Lastly, every vertex of H is exposed to the outside by previous paragraph and every vertex of G except

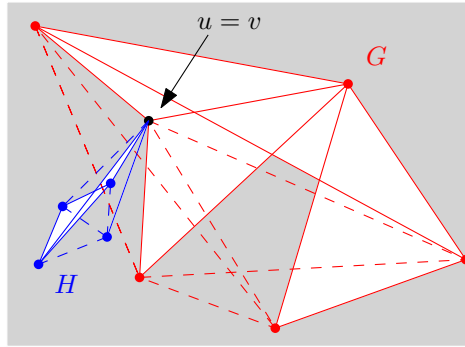


Fig. 6: G is drawn with red color; H with blue. Grey (open) region is an outside obstacle of K .

u is also exposed to the outside because only the subset of A is altered in the obstacle. See Fig. 6 for an example illustration. \square

Lemma 3 *Let H be a graph, v be a vertex of H , A be the set of twins of v , and $G = H \setminus A$. If G has an exposed outside-obstacle representation in which v lies on the convex hull of $V(G)$, then H has an exposed outside-obstacle representation in which all vertices in $A \cup \{v\}$ lie on the convex hull of $V(H)$.*

Proof. Fix an exposed outside-obstacle representation of G such that v is on the convex hull of $V(G)$. We choose ϵ small enough so that a disk D of radius ϵ centered at v doesn't contain any other vertices. We also want ϵ to be small enough so that the outside-obstacle representation where a point for v is replaced by any point in D is still a valid outside-obstacle representation for G . This guarantees that adding A inside D results in a valid outside-obstacle representation for H .

More precisely, let p, q be two intersection points between the convex hull of $V(G)$ and D . We make a slightly bended outwards (for general position assumption) segment C connecting p and q . We then place v and vertices of A on C , say evenly. Since the only part of A is altered from the obstacle, all vertices of H except $A \cup \{v\}$ are exposed to the outside. Since replacing the polygonal curve pvq with C from the convex hull of $V(G)$ still yields a convex region, all vertices of $A \cup \{v\}$ are on the convex hull of $V(H)$ and exposed to the outside. See Fig. 7 for an example illustration. \square

Theorem 1 *If the circumference of a graph G is at most 6, then G has an outside-obstacle representation.*

Proof. Recall that H is a biconnected component of G . Cases 1, 2, and 3 with $\text{circ}(H) < 6$ are described in the main body of the paper. It remains to deal with case 4, that is, with the case $\text{circ}(H) = 6$. In Figs. 8 to 10, we use the following convention: black solid/dashed edges mean determined edges/non-edges; blue dashed edges can be chosen as edges or non-edges freely. Green dashed edges can also be chosen as edges or non-edges but it will be a valid outside-obstacle representation under some conditions.

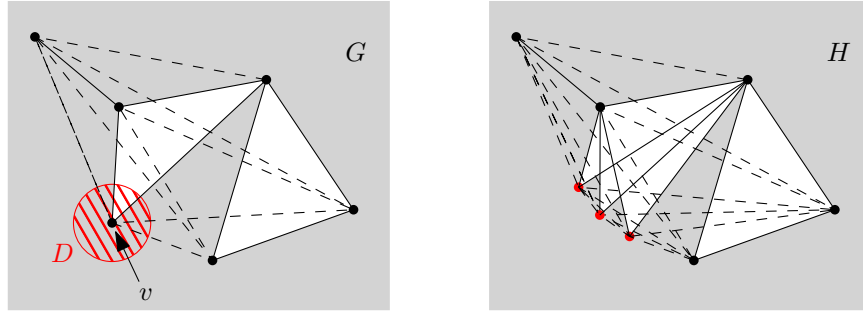


Fig. 7: Grey (open) regions are outside obstacles of G and H respectively; $A \cup \{v\}$ are denoted by red color in H .

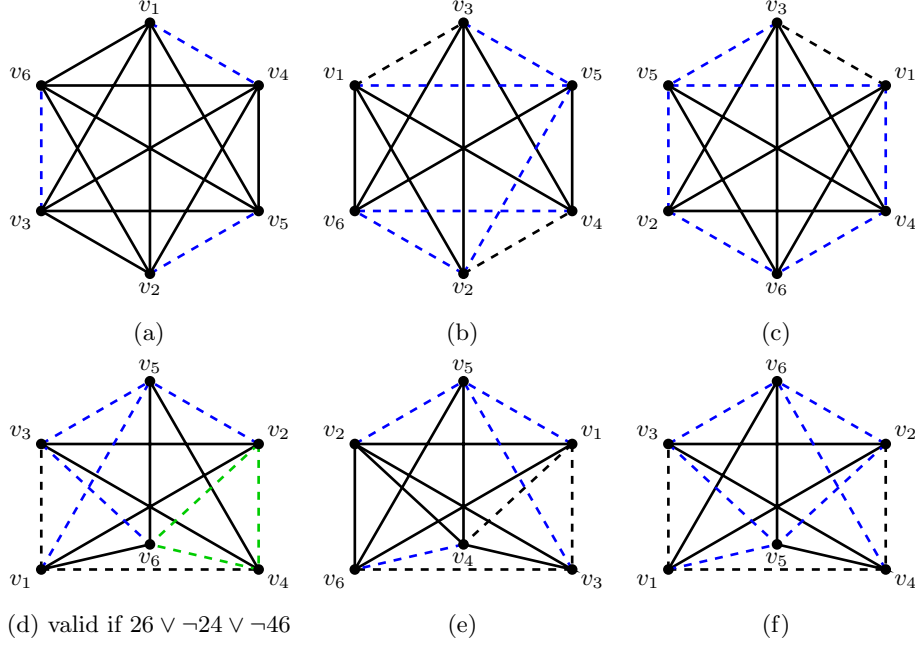


Fig. 8: Outside-obstacle representations of 6-cycle case

Let $C = v_1v_2v_3v_4v_5v_6 \subset H$ be a 6-cycle. If H contains exactly 6 vertices, we can represent it by an outside-obstacle representation. We use a case distinction to prove this. Now to describe the following case distinction, we denote v_1 by 1, and so on. Also, ij means $v_i \sim v_j$ and $\neg ij$ means $v_i \not\sim v_j$. Note that ij is equivalent to ji and $\neg ij$ to $\neg ji$.

If 13, 24, 35, 46, 51, 62, we can use the drawing given in Fig. 8a. Otherwise, without loss of generality, we assume $\neg 13$. We distinguish three cases now:

(a) 14 and 36. If 24, we use the drawing in Fig. 8c. If $\neg 24$, we use the drawing in Fig. 8b.

(b) $\neg 14$ and $\neg 36$. If 24 and 26, we use the drawing in Fig. 8d or in Fig. 8e (depending on the vertex which should be on the convex hull of $V(G)$). Otherwise, without loss of generality, $\neg 24$ (the case $\neg 26$ is symmetric). Then we use the drawing in Fig. 8d or in Fig. 8f.

(c) Without loss of generality, we consider only the case $\neg 14$ and 36. The other configuration (14 and $\neg 36$) is symmetric. If 24, we use the drawing in Fig. 8c. If $\neg 24$, we use the drawing in Fig. 8d or in Fig. 8f.

Now suppose that H contains more than six vertices. We call v_4 (v_5 , v_6 , respectively) an *antipodal* of v_1 (v_2 , v_3 , respectively) and vice versa.

We distinguish five subcases of case (4):

- (i) *There are two vertices in $H \setminus C$ that are adjacent to each other.*

If this is not the case, there are only vertices $x \in H \setminus C$ with $N(x) \subset C$. Then we distinguish the following cases.

- (ii) *There is a vertex in $H \setminus C$ that has only one neighbor in C .*
For the remaining cases we can assume that every vertex in $H \setminus C$ has at least two neighbors.
- (iii) *There is a vertex in $H \setminus C$ that has at least three neighbors in C .*
Therefore, for the remaining cases, we assume that every $x \in H \setminus C$ has exactly two neighbors in C . These neighbors cannot be adjacent on the cycle, as this would imply a longer cycle. So there are only two cases left.
- (iv) *There is a vertex in $H \setminus C$ whose neighbors are antipodals.*
- (v) *All vertices in $H \setminus C$ have two non-antipodal neighbors on C .*

We first show that every vertex $x \in H \setminus C$ is adjacent to at least one vertex in C . For contradiction, suppose not. For two vertices v_i and v_j in C , let C_{ij} and C_{ji} be parts of C such that $C = v_i C_{ij} v_j C_{ji} v_i$. Since x doesn't have any neighbors in C , biconnectivity of H implies that there are vertices $v_i, v_j \in C$ and a v_i - v_j path P of length at least 4 containing x , internally disjoint with $v_i C_{ij} v_j$ and $v_j C_{ji} v_i$. Concatenating P and the longest path among $v_i C_{ij} v_j$ and $v_j C_{ji} v_i$, yields a cycle longer than 6.

We make the following observation before starting the case analysis.

Observation 3 *Let G be a graph with $\text{circ}(G) = k$ and C be a k -cycle in G . If adding an edge ab would create a cycle with a length of more than k , then we cannot add vertices to G such that an a - b path is formed while maintaining the circumference. In particular, we cannot add a vertex x to G that is adjacent to both a and b .*

Case 4(i): There exist vertices $x, y \in H \setminus C$ such that $x \sim y$.

Without loss of generality, $x \sim v_1$. Let H' be a graph obtained by removing twins from H . We show that all maximal v_1 - v_4 paths of H' are internally disjoint.

Observation 1 implies $x \not\sim v_2, v_6$. The same observation shows $y \not\sim v_2, v_3, v_5, v_6$. As H is biconnected, there is a path from y to a vertex of C other than v_1 . This path cannot be longer than 1 because otherwise there would be a longer cycle. This shows $y \sim v_4$. Observation 1 now also implies $x \not\sim v_3, v_5$.

More generally, we claim that every v_1 - v_4 path has length at most 3. For a contradiction, suppose the v_1 - v_4 path P is longer than 3. If P is internally disjoint from $v_1 v_2 v_3 v_4$, then $v_1 P v_4 v_3 v_2 v_1$ forms a cycle longer than 6, so P must contain v_2 or v_3 . Similarly, P must contain v_5 or v_6 , hence there is a path between v_2/v_3 and v_5/v_6 which avoids v_1 and v_4 . This fact contradicts Observation 3 since $v_2, v_3 \not\sim v_5, v_6$.

Let $v_1 a b v_4, v_1 c d v_4, v_1 e f v_4$ be internally disjoint v_1 - v_4 paths. There are at least three of these paths (using the vertices v_2, v_3 or v_6, v_5 or x, y). An edge ac would create a 7-cycle $v_4 f e v_1 a c d v_4$ and an edge ad would create an 8-cycle $v_4 b a d e v_1 e f v_4$, so $a \not\sim c, d$. Same holds for b . In particular, this implies $v_2 \not\sim v_5, v_6$ and $v_3 \not\sim v_5, v_6$. Similarly, we can show that all vertices from different internally disjoint v_1 - v_4 paths (including ones of length 2) are pairwise non-adjacent.

Let $z \in H \setminus C$. If $z \sim v_1, v_4$, z forms an v_1 - v_4 path of length 2, which is too short. If $z \not\sim v_1, v_4$ then there exists a v_1 - v_4 path of length at least 4, which

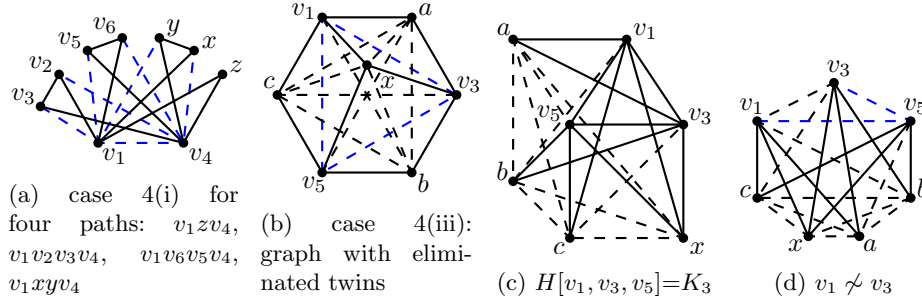


Fig. 9: Cases 4(i) and 4(iii)

we showed impossible. Otherwise, without loss of generality, $z \sim v_1$ and $z \not\sim v_4$. Let $v_1z w v_4$ be a v_1-v_4 path containing z . Since we only need to consider not internally disjoint paths, without loss of generality, let $v_1z' w v_4$ be another v_1-v_4 path. If $z' \sim v_4$ then it would create a v_1-v_4 path of length 4, $v_1z w z' v_4$. It follows that z' is a twin of z . Consequently, all v_1-v_4 paths are internally disjoint after removing twins.

Since we can handle twins using Lemma 3, it's enough to provide an outside-obstacle representation with points in convex position of a graph whose v_1-v_4 paths are all internally disjoint. Place v_1 and v_4 arbitrarily. Draw a half-circle such that v_1 and v_4 lie on its diameter; denote its center by O . Assume there are m disjoint paths and put them in an arbitrary order. Place vertex u , which is on the i -th path, on the half-circle so that $\angle uv_1O = \frac{2i+1}{2m+1}\pi$ if $u \sim v_1$. Otherwise ($u \not\sim v_1, u \sim v_4$), place it so that $\angle uv_1O = \frac{2i}{2m+1}\pi$: see Fig. 9a for an example.

Case 4(ii): There exists a vertex $x \in H \setminus C$ that has only one neighbor in C .

Without loss of generality, $x \sim v_1$. As H is biconnected and using Observation 1, there exists a v_1-v_4 path of length 3 containing x . Therefore this case reduces to case (i).

Case 4(iii): There exists a vertex $x \in H \setminus C$ with at least three neighbors in C .

Without loss of generality, $x \sim v_1$. By Observation 1, $x \sim v_3, v_5$ and $x \not\sim v_2, v_4, v_6$. We make the following two observations.

- (a) Since x plays the same role in the 6-cycle $v_1xv_3v_4v_5v_6$ as v_2 in C , by the same logic as above, $v_2 \not\sim v_4, v_6$. Similarly, $v_4 \not\sim v_6$.
- (b) Assume that there is another vertex $y \in H \setminus C$. If $y \sim v_2$, then $y \not\sim v_4, v_6$ by Observation 3, $y \not\sim v_1, v_3$ by Observation 1, and $y \not\sim v_5$ with a path $v_1v_2yv_5$ and 6-cycle $v_1xv_3v_4v_5v_6$. Since y has at least two neighbors in C , this is a contradiction. An analogous argument holds for $y \sim v_4$ and $y \sim v_6$. Hence, y is adjacent to two or three of the vertices v_1, v_3, v_5 .

Together, (a) and (b) imply that every vertex in $H \setminus \{v_1, v_3, v_5\}$ has two or three neighbors among v_1, v_3, v_5 . Note that when $v_1 \sim v_4$, v_4 is a twin of x . By similar arguments we can exclude $v_2 \sim v_5$ and $v_3 \sim v_6$. It follows that

the graph H' (that is, H after removing twins) is an induced subgraph of the graph in Fig. 9b. In particular, v_2 is either a twin of a or x . Similar statements hold for v_4 and v_6 . Since the obstacle number of a graph is an upper bound for the obstacle number of any induced subgraph, it is enough to provide outside-obstacle representations for the graphs shown in Fig. 9b. If v_1, v_3, v_5 are all adjacent, we can use the outside-obstacle representation depicted in Fig. 9c where all vertices except from v_2 are in convex position. Due to symmetry, we can easily change the drawing so that v_2 is on the convex hull of $V(G)$. If at least one pair of v_1, v_3, v_5 is non-adjacent, we assume $v_1 \not\sim v_3$ without loss of generality and provide the representation in Fig. 9d.

Case 4(iv): There exists a vertex $x \in H \setminus C$ that is adjacent to antipodals.

Without loss of generality, $x \sim v_1, v_4$. An edge v_2v_5 would create the 7-cycle $v_2v_5v_6v_1xv_4v_3v_2$ so $v_2 \not\sim v_5$. Similarly $v_3 \not\sim v_6$.

Assume there is another vertex $y \in H \setminus C$, which is adjacent to v_2 . If $y \sim v_6$, then $v_2yv_6v_5v_4xv_1v_2$ would be a 7-cycle, so $y \not\sim v_6$. Additionally, Observations 3 and 1 imply $y \not\sim v_1, v_3, v_5$, and it follows that $y \sim v_2, v_4$. Edges from v_3 to v_1 or v_5 would create a longer cycle. So $v_3 \not\sim v_5, v_1$ and thus y is a non-adjacent twin of v_3 .

Therefore, every $y \in H \setminus C$ is a non-adjacent twin of v_2, v_3, v_5, v_6 or x and thus by eliminating twins using Lemma 3 we obtain the graph depicted in Fig. 10a. Depending on the additional edges we have to use one of the three different representations depicted in Fig. 10. Note that type a has all vertices in convex position and types b and c have two variants with different vertices on the convex hull of $V(G)$.

Now, if $\neg 14$, we use Fig. 10c. If 14, 13, 24, we use Fig. 10c. Case 14, 15, 46 is symmetric to the previous one. So for the remaining cases at least one of $\neg 15, \neg 46$ and at least one of $\neg 13, \neg 24$ is true. If $\neg 13, \neg 46$ or $\neg 15, \neg 24$, we use Fig. 10b. Without loss of generality, we can assume $\neg 13$ which gives us the configuration 14, $\neg 13$, 46, $\neg 15$, 24. If 26 we can use Fig. 10b, otherwise ($\neg 26$) Fig. 10d.

Case 4(v): All vertices in $H \setminus C$ have 2 non-antipodal neighbors on C .

Let $x \in H \setminus C$. Without loss of generality, $x \sim v_1, v_3$. If v_2 has three neighbors on the cycle $C' = v_1xv_3v_4v_5v_6$, we apply the construction in case (iii) with cycle C' instead of C . Otherwise v_2 has exactly two neighbors (v_1 and v_3) and thus x is its non-adjacent twin. Consequently, every additional vertex is a non-adjacent twin of one of the vertices v_1, \dots, v_6 and thus removing twins using Lemma 3 results in a graph with 6 vertices, which we already handled. \square

Theorem 2 *Any graph with at most 7 vertices has an outside-obstacle representation.*

Proof. If the graph does not have C_7 as a subgraph, we are done by Theorem 1. Otherwise, we have a 7-vertex graph that contains the 7-cycle $v_1v_2v_3v_4v_5v_6v_7$. We consider 15 types to cover all cases.

Type 1 (Fig. 11a): It is clear that the figure is a valid drawing if $\neg 13 \vee \neg 35 \vee 15, \neg 35 \vee \neg 57 \vee 37, \dots$. By moving some vertices depending on the situation, we can make the condition tighter. For instance, Fig. 11b is a valid drawing even

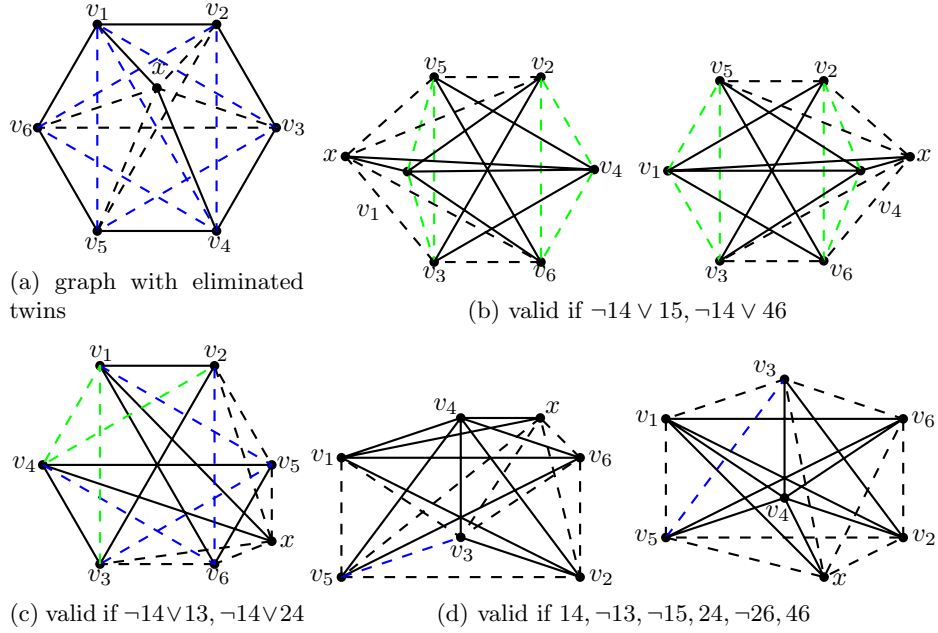


Fig. 10: Case 4(iv)

for 13, 35, $\neg 15$ if $\neg 16 \vee 36$ and $\neg 57 \vee 37$. Hence, we conclude that there is a outside-obstacle representation if there do not exist 3 consecutive edges on the convex hull of $V(G)$ in Fig. 11a. If there are 3 consecutive edges, without loss of generality, v_1v_6 , v_1v_3 , and v_3v_5 are edges. This is only a problem if v_1v_5 and v_3v_6 are non-edges (see Fig. 11c), so we can assume 13, 35, 16, $\neg 36$, $\neg 15$ for the following cases.

Types 2 to 9 cover the case when 47, and types 10 to 15 cover the case when $\neg 47$.

Type 2: 47, 24, 26. When additionally 57, apply Fig. 11d; when $\neg 57$, apply Fig. 11e.

Type 3: 47, 25, 27. Symmetric to type 2.

Type 4: 47, $\neg 24$, $\neg 25$. Fig. 11f.

Type 5: 47, $\neg 26$, $\neg 27$. Symmetric to type 4.

Type 6: 47, 24, $\neg 27$. Fig. 11g.

Type 7: 47, $\neg 24$, 27. Symmetric to type 6.

Type 8: 47, $\neg 24$, 25, 26, $\neg 27$. When additionally 37, apply Fig. 11h; when $\neg 37$, apply Fig. 11i.

Type 9: 47, 24, $\neg 25$, $\neg 26$, 27. Fig. 12a.

Type 10: $\neg 47$, 24, 27. Fig. 12b.

Type 11: $\neg 47$, $\neg 24$, $\neg 27$. Fig. 12c.

Due to symmetry, we can assume $\neg 47$, $\neg 24$, 27 for the rest.

Type 12: $\neg 47$, $\neg 24$, 27, $\neg 25$. Fig. 12d.

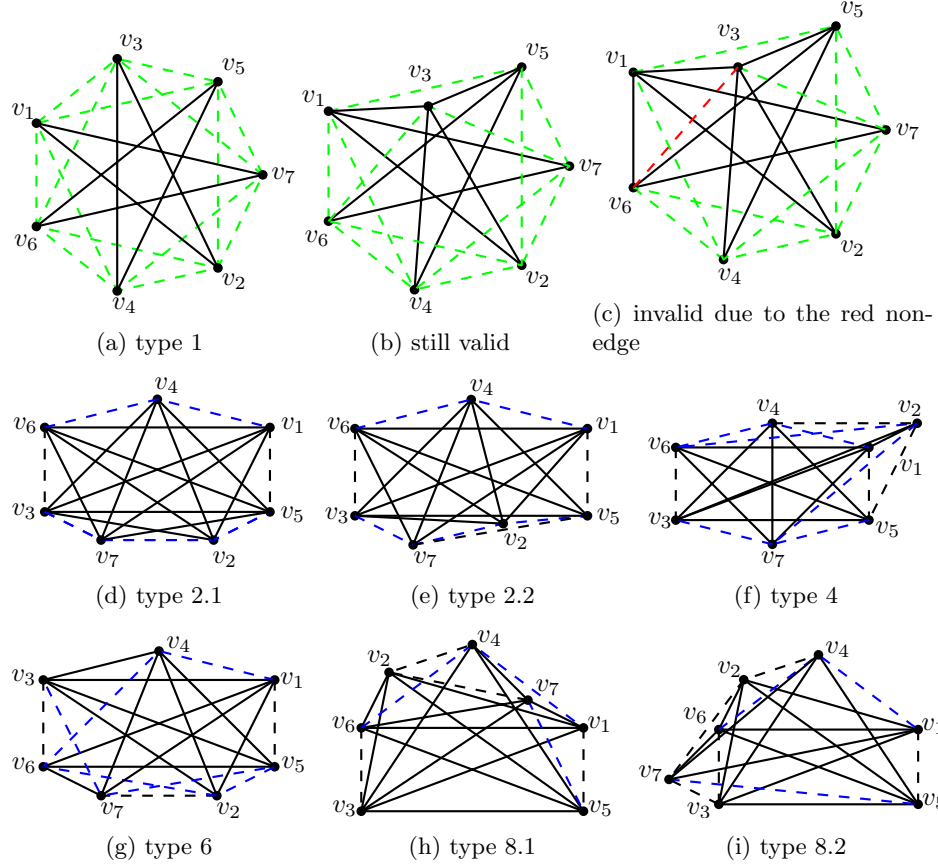


Fig. 11: Outside-obstacle representations of 7-cycle case: types 1 to 8

Type 13: $\neg 47, \neg 24, 27, \neg 57$. Fig. 12e.

Type 14: $\neg 47, \neg 24, 27, 25, 57, 26$. Fig. 12f.

Type 15: $\neg 47, \neg 24, 27, 25, 57, \neg 26$. Fig. 12g.

□

B Missing Proofs of Section 3

This appendix contains the full details showing that B_8 has no inside-obstacle representation, as formalized in Lemma 5. To this end, we first establish some useful properties of graphs with inside-obstacle representations.

Observation 4 *In an inside-obstacle representation of a graph $G = (V, E)$, the vertices on $\text{CH}(V)$ form a cycle.*

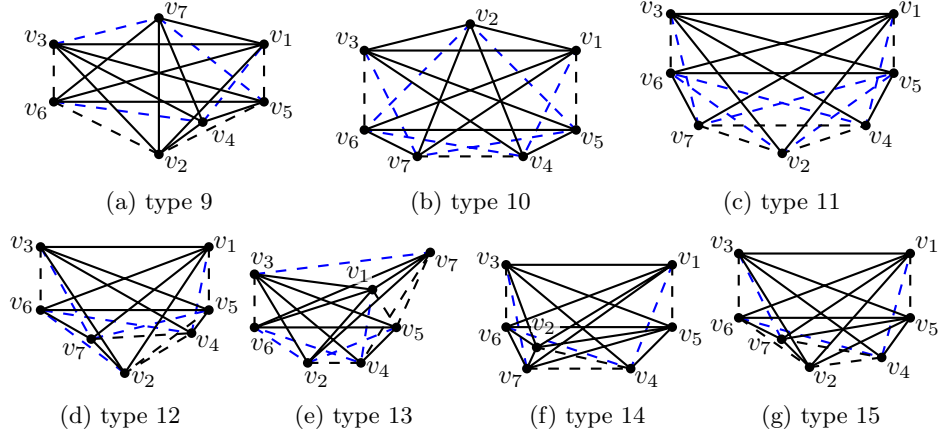


Fig. 12: Outside-obstacle representations of 7-cycle case: types 9 to 15

Observation 5 *In an inside-obstacle representation of a graph G , if G contains a 3-edge induced path $uxyv$ where u and v are on the convex hull of $V(G)$ and x and y are not on the convex hull of $V(G)$, then the line segments \overline{ux} and \overline{yv} do not intersect and the quadrilateral $uxyv$ is convex.*

Proof. Suppose that the line segments \overline{ux} and \overline{yv} intersect, and let z be the intersection point. Let P be a chain on the convex hull of $V(G)$ such that the region bounded by $uzvPu$ contains the line segment \overline{uv} of a non-edge uv . The obstacle should lie inside the region $uzvPu$ due to \overline{uv} . However, the line segment \overline{uy} of the non-edge uy lies completely outside the region $uzvPu$. Thus contradicting $\text{obs}_{\text{in}}(G) = 1$.

Therefore \overline{ux} and \overline{yv} do not intersect, i.e., $uxyv$ forms a non-intersecting quadrilateral. Let P be a chain on the convex hull of $V(G)$ such that the region bounded by $uxyvPu$ contains the line segment \overline{uy} . Notice that, when $uxyv$ is not convex, the line segment \overline{yx} lies outside this region, i.e., contradicting $\text{obs}_{\text{in}}(G) = 1$. Thus, the quadrilateral $uxyv$ is convex. \square

Observation 6 *Let G be a graph which contains the vertices u, v, u', v', x, y such that $uxyv$ and $u'xyv'$ are induced 3-edge paths. In an inside-obstacle representation of a graph G , if u, v, u', v' are on the convex hull of $V(G)$, x and y are not on the convex hull of $V(G)$, and \overline{uv} and $\overline{u'v'}$ intersect, then u, v' or v, u' are not consecutive on the convex hull of $V(G)$. If additionally u, u', v, v' are consecutive, then neither x nor y are contained in the quadrilateral formed by u, u', v, v' .*

Proof. Consider the ray \overrightarrow{ux} , and let p be the intersection point between this ray and the convex hull of $V(G)$. Further, let P be the chain on the convex hull that connects p to v but does not contain u . Similarly, let q be the intersection point of the ray $\overrightarrow{u'x}$ with the convex hull and let Q be the chain on the convex hull that connects q to v' but does not contain u' . By Observation 5, $uxyv$ is convex,

i.e., y is inside the region bounded by $xvPpx$. Similarly, $u'xyv'$ is convex, i.e., y is inside the region bounded by $xv'Qqx$. Thus, the regions $xvPpx$ and $xv'Qqx$ intersect and it follows that P and Q overlap.

Suppose both of u, v' and v, u' are consecutive. Since $u \not\sim v, u' \not\sim v'$, and \overline{uv} and $\overline{u'v'}$ intersect, their order on the convex hull is $uv' \dots vu'$. If x is contained in the quadrilateral $uv'vu'$, then the points p and q occur on the convex hull such that: both p and q are between u and u' (i.e., $u \dots p \dots q \dots vu'$), p is between v' and u' (i.e., $uv' \dots p \dots u'$), and q is between u and v (i.e., $u \dots q \dots vu'$). However, these conditions contradict the fact that the chains P and Q overlap. When x is not contained in the quadrilateral $uv'vu'$, we have the ordering is $uv' \dots vu' \dots p \dots q \dots u$, and, again, P and Q do not overlap. Thus, one of u, v' and v, u' are non-consecutive.

Now suppose that u, u', v, v' are consecutive. Note that, without loss of generality, they are consecutive in that order by the previous paragraph and since $u \not\sim v, u' \not\sim v'$. If x is contained in the quadrilateral $uu'vv'$, we have the convex hull is ordered so that $uu' \dots p \dots q$ and $uu' \dots p \dots v'$ on the convex hull, i.e., causing P and Q to not overlap. Symmetrically, y is also not contained in the quadrilateral. \square

Using the above observations we proceed with the main lemma.

Lemma 5 *The graph B_8 in Fig. 3c has no inside-obstacle representation.*

Proof. Suppose $\text{obs}_{\text{in}}(B_8) = 1$ for contradiction. The following observation, together with Observation 4, greatly restricts the vertices which can occur on $\text{CH}(V(B_8))$.

Observation 7 *Assume that B_8 has an inside-obstacle representation. Then the order of the vertices on the convex hull of $V(G)$ satisfies the following restrictions. If the two vertices $\{v_2, v_4\}$ lie on the convex hull of $V(G)$, then they must be consecutive. The same holds for each of the pairs $\{v_1, v_4\}, \{v_3, v_4\}, \{v_5, v_8\}, \{v_6, v_8\}, \{v_7, v_8\}$, for any maximal subset of $\{v_1, v_2, v_3\}$ occurring on the convex hull of $V(G)$, and for any maximal subset of $\{v_5, v_6, v_7\}$ occurring on the convex hull of $V(G)$.*

Proof. Suppose v_2 and v_4 are both on the convex hull of $V(G)$. If they are not consecutive, then there exist vertices x, y , each distinct from v_2 and v_4 such that x and y occur in distinct chains (P and Q resp.) connecting v_2 and v_4 along the convex hull. Suppose v_5 is also on the convex hull, and that $x = v_5$. Since v_5 is adjacent to all but v_2 and v_4 , x is adjacent to y , but now we see that the line segments $\overline{v_5v_2}$ and $\overline{v_5v_4}$ are separated into disjoint bounded regions by the chord \overline{xy} and the convex hull. Similarly, when v_5 is not on the convex hull, we have the segments $\overline{v_5x}$ and $\overline{v_5y}$ which again, together with the convex hull separate $\overline{v_5v_2}$ and $\overline{v_5v_4}$ into disjoint bounded regions. Hence, in either case, those two non-edges cannot be obstructed by one inside obstacle.

We can similarly use v_6 for $\{v_1, v_4\}$, v_7 for $\{v_3, v_4\}$, v_2 for $\{v_5, v_8\}$, v_1 for $\{v_6, v_8\}$, v_3 for $\{v_7, v_8\}$, v_8 for any subset of $\{v_1, v_2, v_3\}$, and v_4 for any subset of $\{v_5, v_6, v_7\}$. \square

By Observation 7, only two of v_1, v_2, v_3 can be consecutive with v_4 , i.e., at least one of v_1, v_2, v_3, v_4 is not on the convex hull. Symmetrically, at least one of v_5, v_6, v_7, v_8 is not on the convex hull. We now consider the different cases regarding size of the convex hull.

Case 1: $\text{CH}(V(B_8))$ is a 6-gon.

Due to symmetry, we only need to consider 3 cases: when v_1, v_5 are interior to the convex hull, when v_1, v_8 are interior, and when v_4, v_8 are interior. For the first and second case, by Observation 7, v_2, v_3, v_4 are on the convex hull. Thus, v_2, v_3 are consecutive, v_3, v_4 are consecutive, and v_2, v_4 are consecutive, which contradicts $\text{CH}(V(B_8))$ being a 6-gon. For the third case (when v_4, v_8 are interior), we have v_1, v_2, v_3 consecutive and v_5, v_6, v_7 consecutive. Without loss of generality, suppose v_1, v_5 are consecutive. Three orderings are possible on the convex hull: $v_1 v_2 v_3 v_6 v_7 v_5$, $v_1 v_3 v_2 v_7 v_6 v_5$, and $v_1 v_3 v_2 v_6 v_7 v_5$. For $v_1 v_2 v_3 v_6 v_7 v_5$, the non-edge $v_3 v_7$ lies inside the quadrilateral $v_2 v_3 v_6 v_7$ while another non-edge $v_2 v_5$ lies inside the quadrilateral $v_2 v_1 v_5 v_7$. However, these quadrilaterals do not intersect, yielding a contradiction. Note that, $v_1 v_3 v_2 v_7 v_6 v_5$ also has two non-edges $v_3 v_7$ and $v_1 v_6$, occurring within the non-overlapping quadrilaterals $v_3 v_2 v_7 v_6$ and $v_3 v_1 v_5 v_6$ (respectively). We now apply Observation 6 on the two induced paths $v_1 v_4 v_8 v_6$ and $v_2 v_4 v_8 v_5$. This shows that $v_1 v_5$ or $v_2 v_6$ should be not consecutive, invalidating the ordering $v_1 v_3 v_2 v_6 v_7 v_5$.

Case 2: $\text{CH}(V(B_8))$ is a 5-gon.

Without loss of generality, we choose three vertices from v_1, v_2, v_3, v_4 and two vertices from v_5, v_6, v_7, v_8 to be on the convex hull. When v_1 is not on the convex hull, v_2, v_3, v_4 are on the convex hull, which was the case already rejected above. Symmetrically, it is not possible to omit v_2 or v_3 from the convex hull. When v_4 is not on the convex hull, v_1, v_2, v_3 are consecutive by Observation 7. Without loss of generality, the ordering on the convex hull is $v_1 v_2 v_3$. There are 3 possible orderings: $v_1 v_2 v_3 v_5 v_7$, $v_1 v_2 v_3 v_6 v_5$, $v_1 v_2 v_3 v_6 v_7$. First, note that $v_1 v_2 v_3 v_5 v_7$ has two induced paths $v_3 v_4 v_8 v_7$ and $v_2 v_4 v_8 v_5$. Thus, by Observation 6, v_4 and v_8 are inside the triangle $v_1 v_2 v_7$, i.e., a non-edge $v_1 v_8$ lies inside $v_1 v_2 v_7$. However, the region $v_2 v_3 v_5 v_7$ is non-overlapping with $v_1 v_2 v_7$, but contains the non-edge $v_2 v_5$. For the next case (i.e., $v_1 v_2 v_3 v_6 v_5$), we note the two induced paths $v_1 v_4 v_8 v_6$ and $v_2 v_4 v_8 v_5$. Thus, v_4 and v_8 are inside the triangle $v_2 v_3 v_6$. In particular, we have the non-edge $v_8 v_2$ inside $v_2 v_3 v_6$ and the non-edge $v_2 v_5$ inside $v_2 v_6 v_5 v_1$. Finally, for $v_1 v_2 v_3 v_6 v_5$, the two induced paths $v_1 v_4 v_8 v_6$ and $v_3 v_4 v_8 v_7$ fail the condition in Observation 6.

Case 3: $\text{CH}(V(B_8))$ is a 4-gon.

First, we consider the case when three vertices from v_1, v_2, v_3, v_4 and one vertex from v_5, v_6, v_7, v_8 is on the convex hull. As before, the three vertices are v_1, v_2, v_3 . Without loss of generality, we assume $v_1 v_2 v_3$ is the order on the convex hull, i.e., the forth vertex must be v_5 . However, now either $v_1 v_2 v_6 v_5$ or $v_3 v_2 v_6 v_5$ forms a *dart*, so it cannot be represented by 1 obstacle (see Fig. 4 in [1]).

We now have 2 vertices from v_1, v_2, v_3, v_4 and 2 from v_5, v_6, v_7, v_8 on the convex hull. Suppose v_4 is interior and without loss of generality, v_1 and v_2 are on

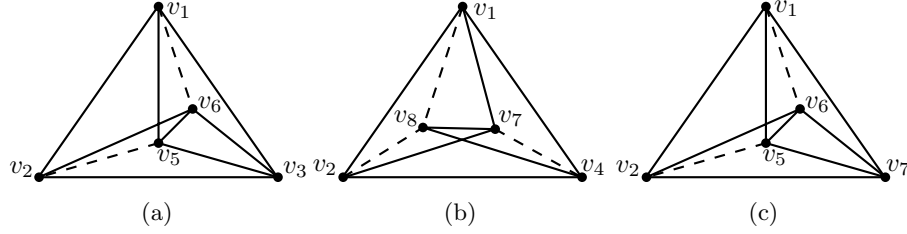


Fig. 13: When $\text{CH}(V(B_8))$ is a 3-gon

the convex hull. There are 3 cases to consider: $v_1v_2v_6v_5$, $v_1v_2v_6v_7$, $v_1v_2v_7v_5$. Since there are only 4 vertices on the convex hull, it suffices to find two induced paths for Observation 6 to provide contradiction. For the first case, we find $v_1v_4v_8v_6$ and $v_2v_4v_8v_5$. For the second case, we observe an induced path $v_1v_4v_8v_6$ and by Observation 5 v_4 and v_8 lie on the same side of v_1v_6 . Hence, the non-edges v_4v_7 and v_2v_8 lie inside the non-overlapping regions $v_1v_4v_8v_6v_2$ and $v_1v_4v_8v_6v_7$ (respectively). For the third case, we apply the same logic but starting from the induced path $v_2v_4v_8v_5$.

Finally, we have v_4 on the convex hull and suppose without loss of generality, that v_1 is as well. This means the convex hull is either $v_1v_4v_8v_5$ or $v_1v_4v_8v_7$. By symmetry, it suffices to consider $v_1v_4v_8v_5$. Here we again observe the two forbidden induced paths: $v_1v_2v_6v_8$ and $v_4v_2v_6v_5$.

Case 4: $\text{CH}(V(B_8))$ is a 3-gon.

Without loss of generality, the convex hull either consists of 3 vertices from v_1, v_2, v_3, v_4 or 2 vertices from v_1, v_2, v_3, v_4 and 1 vertex from v_5, v_6, v_7, v_8 . Up to symmetry, it suffices to consider 3 cases: $v_1v_2v_3$, $v_1v_2v_4$, $v_1v_2v_7$.

When the convex hull is $v_1v_2v_3$, we note the induced 4-cycle $v_1v_2v_6v_5$. Suppose $\overline{v_1v_6}$ and $\overline{v_2v_5}$ intersect, i.e. $v_1v_2v_6v_5$ is a convex quadrilateral. Since the region $v_1v_2v_6v_5$ contains a non-edge v_1v_6 and v_1v_8 , v_5v_4 are non-edges, v_4 and v_8 lie inside $v_1v_2v_6v_5$. However, by Observation 6, the two induced paths $v_1v_4v_8v_6$ and $v_2v_4v_8v_5$ are forbidden. Consequently $\overline{v_1v_6}$ and $\overline{v_2v_5}$ do not intersect. However, in this case, v_1v_6 and v_2v_5 occur in disjoint bounded regions as depicted in Fig. 13a.

When the convex hull is $v_1v_2v_4$, we also observe an induced 4-cycle: $v_2v_7v_8v_4$. Suppose it forms a convex quadrilateral. Since v_3v_7 and v_5v_4 are non-edges, they lie inside $v_2v_7v_8v_4$. However, again, by Observation 6, we have forbidden induced paths $v_2v_3v_5v_8$ and $v_4v_3v_5v_7$. If $\overline{v_2v_8}$ and $\overline{v_7v_4}$ do not intersect, then v_4v_7 and v_2v_8 occur in disjoint bounded regions as depicted in Fig. 13b.

When the convex hull is v_1, v_2, v_7 , we conclude that $\overline{v_1v_6}$ and $\overline{v_2v_5}$ do not intersect similarly to the first case. However, again, similarly to the first case, we see that when $\overline{v_1v_6}$ and $\overline{v_2v_5}$ do not intersect, v_1v_6 and v_2v_5 occur in disjoint bounded regions as depicted in Fig. 13c. \square

C Missing Proofs of Section 4

In this appendix we describe how to use the NP-hardness of the outside-obstacle sandwich problem to show NP-hardness for both the single-obstacle sandwich problem and the inside-obstacle sandwich problem. To this end, we first make an observation.

Observation 8 *Let G be any graph, and let G_{in} be a graph with $\text{obs}_{\text{in}}(G_{\text{in}}) = 1$, but $\text{obs}_{\text{out}}(G_{\text{in}}) > 1$. If G^* is the disjoint union of G_{in} and G (i.e., $V(G^*) = V(G_{\text{in}}) \cup V(G)$ and $E(G) = E(G_{\text{in}}) \cup E(G)$), then the following properties hold:*

1. $\text{obs}_{\text{out}}(G^*) > 1$.
2. *In every inside-obstacle representation of G^* , the point set of G is contained inside the convex hull of the obstacle, i.e., any single-obstacle representation of G^* contains an outside-obstacle representation of G .*

Proof. Property 1 is clear since any outside-obstacle representation of G^* would certainly contain an outside-obstacle representation of G_{in} , contradicting $\text{obs}_{\text{out}}(G_{\text{in}}) > 1$.

For Property 2, suppose that G has an inside-obstacle representation and let P be the obstacle. Notice that P must be strictly contained within the convex hull of the point set of G_{in} (otherwise we would have an outside-obstacle representation of G_{in}). In particular, P is contained in a region whose boundary consists of (parts of) edges of G_{in} . Now consider any vertex v of G , and suppose for a contradiction that v is not placed within the convex hull of P . This means that there is a line ℓ such that

- ℓ intersects the boundary of P ,
- the interior of P is contained in one open half-plane h^+ defined by ℓ , and
- v is contained in the other open half-plane h^- .

However, as P is an inside obstacle of G_{in} , there must be a vertex u of G_{in} that is contained in h^- , contradicting the fact that the obstacle P must intersect the line \overline{uv} . \square

From this observation, we can extend the NP-hardness of the outside-obstacle sandwich problem to both the inside-obstacle sandwich problem and the single-obstacle sandwich problem.

Corollary 1 *The inside-obstacle graph sandwich problem and the single-obstacle graph sandwich problem are both NP-hard. These problems remain NP-hard even when restricted to sandwich instances (G, H) where G is connected.*

Proof. Let (G, H) be an instance of the outside-obstacle graph sandwich problem. Recall that, for the graph B_{11} (given in Fig. 3b), we have $\text{obs}_{\text{in}}(B_{11}) = 1$ and $\text{obs}_{\text{out}}(B_{11}) > 1$. From the pair $(G \cup B_{11}, H \cup B_{11})$, we make $3|V(G)|$ instances $\mathcal{I}_1, \dots, \mathcal{I}_{3n}$ of the single-obstacle sandwich problem where each instance $\mathcal{I}_i = (G \cup B_{11} \cup \{u_i v_i\}, H \cup B_{11} \cup \{u_i v_i\})$ is formed by adding a single edge $u_i v_i$

connecting a vertex u_i of G to a vertex v_i of B_{11} . Due to the symmetry in B_{11} , there are at most $3n$ non-isomorphic ways to add such an edge.

We now claim that (G, H) has a solution to the outside-obstacle sandwich problem if and only if some single-obstacle sandwich instance \mathcal{I}_i has a solution. Moreover, a solution to \mathcal{I}_i is always an inside-obstacle representation. This proves the statement of the corollary. It remains to show that our claim holds.

For the forward direction, let G' be a solution to the outside-obstacle sandwich instance (G, H) . We can place G' “inside” the obstacle of the inside-obstacle representation of B_{11} (e.g., the one depicted in Fig. 3b) to obtain an inside-obstacle representation of $B_{11} \cup G'$. Furthermore, we can make a thin “tunnel” into the obstacle so that we realize precisely one edge connecting G' and B_{11} . This provides an inside-obstacle representation of one of the instances \mathcal{I}_i .

For the reverse direction, consider an instance $\mathcal{I}_i = (G \cup B_{11} \cup \{u_i v_i\}, H \cup B_{11} \cup \{u_i v_i\})$ that has a solution G' . Recall that $u_i \in V(G)$ and $v_i \in V(B_{11})$. Let P be the corresponding obstacle. Note that, by Observation 8, G' also provides a single-obstacle representation of $G' \setminus \{u_i\}$ using P , and this must be an inside-obstacle representation. Moreover, $G'[V(G) \setminus \{u_i\}]$ is contained in the convex hull of P . In particular, P is an outside obstacle of $G'[V(G)]$, and P is an inside obstacle of G' . \square

D Inside-Obstacle Number 2, but Outside-Obstacle Number 1

Since inside-obstacles cannot pierce the convex hull of a drawing, every graph with an inside-obstacle representation must either be complete or contain a cycle. This appendix introduces a non-trivial graph (i.e., containing a cycle) that has outside-obstacle number 1 but inside-obstacle number greater than 1. It is trivial that every 4-vertex graph G which contains a cycle has $\text{obs}_{\text{in}}(G) = 1$. On the other hand, we will show that there is a unique 5-vertex graph with $\text{obs}_{\text{in}}(G) > 1$ and, as such, the 4-vertex observation is tight.

Theorem 8 *Among all 5-vertex graphs, $K_{2,3}$ is the unique graph that contains a cycle, has a outside-obstacle number 1, and inside-obstacle number greater than 1.*

Proof. Let $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$ be a bipartition of $K_{2,3}$. By Theorem 2, $K_{2,3}$ has an outside-obstacle representation. To prove that $\text{obs}_{\text{in}}(K_{2,3}) > 1$, we assume $\text{obs}_{\text{in}}(K_{2,3}) = 1$ for contradiction. Since the convex hull should form a cycle in an inside-obstacle representation by Observation 4, without loss of generality, we assume the cycle $u_1 v_1 u_2 v_2$ is the convex hull. By placing v_3 inside quadrilateral $u_1 v_1 u_2 v_2$, we notice that non-edges $v_1 v_3$ and $v_2 v_3$ lie inside the different bounded regions so a single inside obstacle cannot block both.

To prove uniqueness, let G be a 5-vertex graph not isomorphic to $K_{2,3}$. It is enough to provide an inside-obstacle representation for the connected graphs with no leaves. Since G doesn’t have a leaf and isn’t isomorphic to $K_{2,3}$, G

contains a 5-cycle C . To make an inside-obstacle representation of G , we place the points of C as a regular 5-gon. Notice that the diagonals of C make a star-shape inside the 5-gon which, in turn, provides an inner 5-gon P where each side corresponds to a diagonal of C . We can use P as an obstacle by simply bending each side outward when the corresponding diagonal of C is a non-edge.

This finishes the proof of Theorem 8. \square